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THE MATHEMATICS STUDENT

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TIME-MEASUREMENT ANCIENT, MEDIÆVAL AND MODERN

BY

SUKUMAR RANJAN DAS, *Delhi*.

Long before the physical and philosophical notion of Time was developed, it was found necessary to have a standard measurement of Time for all practical purposes—religious as well as secular. There arose in all the ecclesiastical schools of the ancient nations the necessity for instructing some member or group of the priestly order in the process of computing the dates of the religious festivities. (1) The problem was universal and was not confined to any particular religious sect. Since most early religions were connected with sun worship or with astrology, work somewhat similar to that of preparing the computus was needed in all religious organizations—Hindu, Greek, Egyptian, Chinese, Babylonian, Hebrew, Mahammadan and ancient Roman. (2) Among all the nations, the fundamental periods of Time, the day, the month, the year, are the same, the variations occurring in them being principally in the arrangement of the days to form months and years; in the subdivisions of the day; in the times to be reckoned as the commencement of the day, whether at midnight, sunrise, or noon; in the sub-divisions of the year into months, differing from each other as to the number of days of each; in the various kinds of months to form the year, and the like. There have been in all nations certain difficulties experienced as to the time when the year should be reckoned to begin, and in the consequent arrangement of the months and seasons, so that they should recur at regular intervals.

Naturally the revolution of the moon and the sun served for fixing the standard by which Time was to be measured. The ancient thinkers were struck with the daily appearance and disappearance of the moon and the sun. Consequently the movement of the moon or

(1) The Discovery of Time by J. T. Shotwell, *Journal of Philosophy, Psychology and Scientific Methods* XII.

(2) Egyptian papyrus of the beginning of the Christian Era is still extant.

the sun was chosen to fix the measure of Time. The early religion of the ancient nations reveals an intimate knowledge of the times and seasons, and there was from the earliest times an attempt to prepare a calendar setting forth the order in which the rites and ceremonies of the nations should be observed. This calendar, in the earlier periods, was of an imperfect character, which led to methods afterwards adopted for its improvement, generally with a view to its adaptation to religious rather than to secular uses.

The ancient Hindus prepared their calendar mainly for sacrificial purposes, and the performance of various sacrifices facilitated the maintenance of the calendar. When the course of sacrifices was completed, it was found that the year had also run its course, and the sacrifice and the year, therefore, became synonymous terms. There are several sacrificial hymns in the Rigveda (3000 B.C.), which show that the sacrificial ceremonies must have been considerably developed; and as no sacrificial system could be developed without the knowledge of months, seasons and the year, it will not be too much to presume that in the Vedic times there must have existed a calendar to regulate the sacrifices. It is difficult to determine the exact nature of this calendar, but a study of the sacrificial literature would show that the phases of the moon, the changes in the seasons, and the southern and northern courses of the sun were the principal landmarks in the measurement of Time in those early days.[3] The difficulties experienced by the Hindus in adjusting their calendar occasioned repeated changes of their system. At one period the motion of the moon was taken as its foundation, and the lunar month was formed to agree with the phases of the moon. The ancient Hindus found that the moon totally disappeared one night and again became full and round another night; they called these phases new and full moon and further observed that from one new moon to another or from one full moon to another the sun rises thirty times. Hence one lunar month became equivalent to thirty days. Then a change in the calendar took place, and a solar month was formed, constituted so as to be reckoned by the time the sun, in its progress, remained in each sign of the solar Zodiac. Another change followed, efforts being made to reconcile the two previous systems, in which each kind of month preserved its original character, the solar month being reckoned in ordinary civil days, and the lunar months measured by lunar days, each being one-thirtieth part of a synodic period, the time elapsing between two conjunctions of the sun and the moon.

(3) Orion by B. G. Tilak, page 11.

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The result of these efforts was the formation of the luni-solar year, either in civil days or in lunar days. The Hindus further observed that the star that rose or set at sunrise one day, would not do so after the lapse of several days. They concluded that the sun like the moon moved among the stars in the heavens and that the sun took twelve months to complete this course. Thus according to this calculation a year contains twelve months. The Hindus made several changes regarding the beginning of the day. The Vedic and the Paurāṇic literature maintained that the day began with sunrise: but different theories on this question were advanced by the later Hindu writers. Āryabhaṭṭa maintained that the beginning of the day is to be reckoned from sunrise at Laṅkā. Varāhamihira held that the day begins from midnight. In fact, there is mention of four kinds of day-beginnings, namely, from sunrise, from midnight, from midday and from sunset.[4] The division of the calendar into a year, months, weeks and days evolved gradually but the system was almost complete in the Vedic and the Paurāṇic ages. Slight modifications were introduced later on by the Hindu astronomers and this modified calendar has been since then in vogue in India. The Hindus at a very early date invented the sundial (Gnomon measuring twelve fingers) to measure the sun shadow from which time was determined. The observation of the increase and decrease of the shadow of a tree must have struck them with the idea of a Gnomon. But though the Gnomon was sufficient to measure the time during the day, it was not possible to use it after dusk. The water-clock or clepsydra (a metal bowl floating in a vessel of water where the amount of water that measures a nāḍikā or 24 minutes is given) was, therefore, invented by the ancient Hindus. In fact, they became so skilful in the use of water-clock or clepsydra that they could find out the exact time at a mere look at the instrument. There was also another instrument called Yaṣṭi or staff to measure the time from sunrise or the time after midday.[5]

The Chaldeans knew the length of the year as 365 days 6 hrs. 11 m., but used both the lunar month and the lunar year for civil purposes. They divided both the natural day and the natural night into twelve hours each. They invented the sundial and the water-clock to measure time, the former was used during the day and the latter during the night. For astronomical purposes they divided the day into twenty-four equal hours. They also used very early a fourth of

(4) *Vide* a paper on Hindu Calendar by Sukumar Ranjan Das, *Indian Historical Quarterly*, September, 1928.

(5) *Siddhānta Śiromaṇi*, Golādhyāya, Chapter XI, verses 28-30.

a month (as a convenient division of Time; this was probably half of the half month as was customary in ancient world.[6]

Earlier than 2000 B.C., the Chinese attempted to prepare a calendar. But the system at first changed with each emperor. Under the emperor Yan (c. 2357 B.C.—c. 2258 B.C.) an effort was made to establish a scientific calendar for the whole country, and possibly this was done even earlier, under the emperor Huang-ti (c. 2700 B.C.)[7] There is evidence that according to a decree of Wn-wang (1122 B.C.) the day was arranged to have begun with mid-night, although before this under the Shang dynasty (1766-1122 B.C.) the day began at noon. In the Chinese calendar the civil day has twelve hours and the middle of the first hour is midnight. These hours which were called Shi in the Chinese language are each 120 European minutes in length. Each hour is again divided into eight parts called Khe which is equal to ¼ of an European hour. Each of these parts is then divided into 15 Fen, one fen being equal to one European minute. Each fen is again divided into 60 miao, one miao being equal to one European second. At present also the American clock is becoming common in China. The Chinese days were named in such a way as to give seven-day periods and the month began with new moon.

As early as the 14th century B.C. the Egyptians recognised the value of a fixed year, but the changing one was so strongly implanted in the religious canons of the people that it could not be possibly given up. The fixed year was used to the extent of a division into three seasons, regulated by the river,—the Water Season, the Garden Season, and the Fruit Season (namely, June 21 to October 20; October 21 to February 20; February 21 to June 20). These were easily determined by the Temple observers. They were left with the determination of the seasons and were the principal calendar-makers. From the temple, too, came the announcements of the turn in the rise or fall of the river, the nilometer being under the observation of the temple-priests.[8] In the ancient Egyptian calendar, the business day included the night, the natural day and night being each divided into twelve hours, these hours varying in length with the season. The civil day seems to have commonly begun at sunset. But it is said by Pliny that the priests began their day at midnight. In later times the civil day began at noon, and was divided into twenty-four

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(6) "Resume de Chronologie Astronomique," *Memoire de l' Academie des Sciences*, XXII, pp. 209-476, by J. B. Biot.

(7) History of Mathematics in China and Japan by Mikami, pp. 5-45.

(8) For the details *vide* The Evolution of Calendars by M. B. Cotsworth, Washington, 1922.

equal hours. This was also the case with Ptolemy the astronomer (c. 150 B.C.). In the native Egyptian calendar each month except the last (Mesori) contained thirty days; five days used to be added to Mesori so as to make the year equal to 365 days. This gave an error of ¼ of a day. The year was a changing one and came back to its original position with respect to the heavenly bodies once in 4×365 common years, or 1460 years (1461 Egyptian years). The year began with the first day of month Thoth, the God, who was supposed to introduce the calendar and numbers into Egypt.

After Egypt became a Roman province (c. 30 B.C.) the Alexandrian calendar, including the fixed year, was introduced, although the varying year remained until the fourth century A.D. The Alexandrian system was used in the first half of the seventh century till 638 A.D. when Alexandria yielded to Mahamadan conquest. There was then a change in the calendar except in Upper Egypt. In latter days when the French obtained brief control of Egypt in 1798 the European system was used in Egypt side by side with the Mahammadan system.

The Athenian calendar[9] followed the Egyptian in beginning the new day at sunset and in dividing both day and night into twelve hours. The seven-day week was not used. However, the lunar month was divided into three parts. The first part was equal to ten days numbered in order, the "fifth day of the beginning of the month" being the fifth. Then followed nine days, numbered as before, but with the designation "over ten", i.e., "one over ten" and so on to "twenty". Then to the end of the month the numbers were "one over twenty" and so on. These days were also used to be numbered backwards from the end of the month. In the popular calendar the month began with the new moon, and twelve of these months made 354 days, requiring the insertion of a new month every three years. This was called the second month of Poseideon, known as Poseideon II. In 432 B.C. Meton constructed a nineteen-year cycle in which the third, fifth, eighth, eleventh, thirteenth, sixteenth and nineteenth years used to contain the extra month. Nineteen years contained two hundred and thirty-five months and were equal to $6939\frac{3}{4}$ days. These months as arranged, however, contained $6940\frac{2}{3}$ days. In 325 B.C. Callippus modified it to include four nineteen-year cycles. 4×19 years $= 76$ years $= 940$ months. The months were 29 or 30 days and therefore 940 months were equal to 27759 days. Still

(9) History of Mathematics, Vol. II by D. E. Smith, pp. 660-664.

later in 150 B.C. Hipparchus suggested the use of four of the cycles of Callippus. But none of the last two calendars ever came into popular use.

The Romans considered the planets as ruling over one hour of each day, in the following order, beginning with the first hour of Saturday: Saturn, Jupiter, Mars, Sun, Venus, Mercury, Moon. At that time the sun and moon were placed among the wanderers (planets). Taking Saturn for the first hour of Saturday and counting the hours forward we find that the second hour is ruled by Jupiter and so on to the twenty-fourth, which is ruled by Mars. Then the next hour, the first of Sunday, is ruled by the Sun, the first of the next day by the Moon and so on. Thus the days of the week were named after the ruling planets of their first hours. Hence we have Saturn's day, Sun's day, Moon's day, Mar's day (French Mardi) Mercury's day (French Mercredi), Jupiter's day (In the northern lands, Thor's day), Venus's day (Frigg's day, Frigg was the goddess of marriage). The oldest of the Roman calendars is attributed to Romulus, the founder of Rome. The year, then, probably consisted of ten months of varying lengths, or of 304 days. The year began with March. Numa Pompilius (715-672 B.C.) added two other months, January and February, and his year was probably lunar. In the fifth century B. C. the Decemvirs decreed a solar year, the regulation of which was left in the hands of the priests. The calendar was so mismanaged that by the time of Julius Caesar each day was eighty days out of its astronomical place. Radical measures were, therefore, necessary for the reform of the calendar and Caesar decreed that the year 46 B.C. should have 445 days and that thereafter the year should consist of 365 days with a leap year every fourth year. The difficulty arose thus; the ordinary civil year contains an exact number of days, viz. 365, but the time taken by the sun to complete a revolution in the ecliptic is about $365\frac{1}{4}$ days and the exact interval between the successive vernal equinoxes is 365 days 5 hrs. 48 min. 45·5 sec. which is the Tropical Year. Therefore, by taking the civil year as 365 days, there is an error compared with the tropical year of 5 h. 48 m. 45·5 sec., which in four years amounts to 23 h. 15 m. 2 sec., or very nearly a day. If this error was not corrected, the result would be that the dates of the equinoxes and solstices would be later by one day every four years. The first exact attempt at approximating the length of the civil to that of the tropical year was made by Julius Caesar. It was then agreed that an additional day should be given to every fourth year, which was to contain 366 days. Caesar followed the following plan

to determine the names of the months and the number of days to constitute each month:—

	Names of the months	Number of days
1.	Martius	31
2.	Aprilio	30
3.	Maius	31
4.	Junius	30
5.	Quintilis	31
6.	Sextilis	31
7.	Septembris	30
8.	Octobris	31
9.	Novembris	30
10.	Decembris	31
11.	Januarius	31
12.	Februarius	28

Here the year was made to begin in March. An account is also found for the origin of the nomenclature of September (seventh month), October, November, and December. In his original plan Caesar caused every alternate month, beginning with March, to have 31 days, the others having 30 days, except that February received its 30th day once in four years, otherwise it had 29 days. Later Julius Caesar decreed that the year should begin with January. Finally, during his life-time he changed the name of Quintilis, the month in which he was born, to that of Julius, after his own name. He also changed the number of days in certain months and the result was the present Julian Calendar. After the death of Julius Caesar and in the second year of his calendar a further confusion arose, apparently through a misunderstanding on the part of the priests as regards the proper date for leap year. This was corrected by Augustus, and in his honour the name of Sextilis was changed to bear his name. Since then the Julian calendar was in vogue till it was further reformed by Pope Gregory XIII in 1582. According to the Julian Calendar a correction was made of one day in four years. But one day, or 24 hours, was in excess of 23 hrs. 15 min. 2 sec. by about 45 minutes. Thus the correction by means of leap year led to a new but very much smaller error of about 45 minutes in four years, or an average of rather more than eleven minutes each year. This error, in 400 years, would amount to nearly three days. Hence was necessary the Gregorian correction to the Julian Calendar adopted by Pope Gregory, according to which each year which is a multiple of 100, such as 1700, 1800, 1900, which by the Julian Calendar, are

leap years, should be ordinary years, with the exception of those years in which the number of the century is divisible by 4 without remainder, such as 2000, 2400, which should remain leap years. This arrangement evidently makes the required correction of three days in 400 years. Even with the Gregorian correction there is still a very small error, which, however, would amount to not more than a day in 20,000 years. The Gregorian correction was not adopted in England until the year 1752, when the accumulated error, as compared with the corrected calendar, amounted to eleven days. Eleven days of 1752 were, therefore, skipped, the 2nd of September being called the 13th. The Julian Calendar was used by the Greek Catholics, including the Russians until the world war of 1914-1918, the dates at that time differred by thirteen days from those of the calendar of Western Europe where the Gregorian correction was adopted.

It is interesting to note that in the early centuries the year in the Christian calendar usually began with April in the East of Europe[10] and with March in the West, although sometimes with the Feast of the Conception, Christmas day, Easter, or Ascension day, or at other times according to the orders of the Popes. In Spain until the sixteenth century and in Germany from the 11th century, March 1 and March 25 were the favourite days to make the year-beginning, although Advent Sunday (the fourth Sunday before Christmas) has generally been recognised as the beginning of the ecclesiastical year. March 1 was generally used in mediaeval France for beginning the year. The same was the custom in Oriental Christendom, and in Venice until 1797. March 25 was used by the mediaeval Pisans and Florentines for year-beginning. In Italy Pope Innocent XII decreed that the year should begin on January 1, beginning with 1691, as Philip II had done for the Netherlands in 1575, and as Julius Caesar had done before the Christian Era. Most of the Italian states adopted January 1 in 1750. England adopted it in 1752. [11]

We may mention here that there is evidence of several changes of the year-beginning in the Hindu calendar. In the early Vedic times the year began when the sun was in the vernal equinox. Later on, the commencement of the year was changed from the vernal equinox to the winter solstice. It is difficult to ascertain definitely the time of the change. Now to understand this change

(10) The Byzantine Calendar began with September 1.

(11) History of Mathematics, Vol. II page 651 and pages 661-670, by D. E. Smith.

in the beginning of the year, it is necessary to remember that the solar year was sidereal and not tropical in the case of the Hindu Calendar, and that the great object of the calendar was to ascertain the proper time of the seasons. This necessitated a change in the beginning of the year, every two thousand years or so, to make it correspond with the cycle of natural seasons. The difference between the sidereal and the tropical year is 20·4 minutes, which causes the seasons to fall back nearly one lunar month in about two thousand years, if the sidereal solar year be taken as the standard of measurement. Therefore, the beginning of the year was twice altered owing to the precession of the equinoxes. The third change in the year-beginning was introduced at the time of the Vedanga Jyotisa (about 500 B.C.), when the seasons had further receded by a fortnight and the beginning of the month was altered from the full-moon to the new-moon during this period. The next change was introduced and put into effect by the astronomer Varahamihira, in the beginning of the sixth century A. D. and this last system is even now being used in India.[12]

The Jews began their day at sunset, their week on Saturday night (i.e., when their holyday ends and Sunday begins) and their year with Fishi 1 (the first new moon after the autumnal equinox).[13] Their calendar was lunar. In the ancient Maya civilization the year began with the Winter Solstice, was divided into eighteen months and was entirely independent of astronomical considerations. Scholars assert that their calendar goes back to the thirty-fourth century B.C. The Mahammadans began their day with sunset, divided both day time and night time into twelve hours, the length of the hour varying with the season; their week began on Sunday, their month with the new moon, and the year was purely lunar of 354 or 355 days.

When the year-beginning was fixed it was found necessary to determine the date from which the years were to be numbered. In India the Saka Era which is 78 years behind the Christian Era is generally followed, i.e., the date of accession of a famous Saka King is taken to number the years. In Europe, following the Roman custom, the years in the early centuries were dated from the accession of the emperor or consul until the abbot Dionysius Exignus (533 A.D.) arranged the Christian calendar in such a way

(12) Seasons and Year-beginnings of the Hindus by Sukumar Ranjan Das.

(13) Elements of the Jewish and Mahammadan Calendars, by S. B. Bunaby, London 1901.

as the supposed date of the birth of Christ was generally taken for the beginning of the era. This era was adopted in Rome in the sixth century. The Mahammadans begin their era from the date of birth of their Prophet, Mahammad. To get the Christian year from the hejira (Mahammadan year), we add 622 to 97 per cent of the number of the hejira year, i.e., 1300 A. H $= \frac{97}{100} \times 1300 + 622 = 1883$ A.D.

In the early days of the French revolution an attempt was made to impose a new calendar upon the country, partly as a protest against the Christian Church. The calendar was to begin with the autumnal equinox on September 22, 1792. The months of this calendar were named according to natural conditions.

Of the chief divisions of time the most obvious one was the day. This was, therefore, the primitive unit in the measurement of time and the one which for many generations must have been looked upon as unvarying. As the human race developed, however, various kinds of day were distinguished. First from the standpoint of invariability is the sidereal day, the interval of time taken by the fixed stars to complete a revolution round the pole, namely 23 hrs. 56 m. 4·09 seconds of our common time. But from the standpoint of the casual observer, however, first is the true solar day, the length of time between one passage of the sun's centre across the meridian and the next passage. This latter day varies with the season, the difference between the longest and shortest day being 51 seconds; but for common purposes the solar day sufficed for thousands of years and the sundial was frequently used to measure the length of time. As clocks became perfected a third kind of day came into use, the artificial mean solar day which is the average of the variable solar days of the year and equal to 24 hrs. 3 m. 56·56 seconds of sidereal time. In addition to these general and obvious kinds of day there are others which have been mentioned by the writers on chronology. With the Babylonians the day began at sunrise; with the ancient Hindus it began at sunrise or midday or midnight, or sunset at various periods, but most generally from sunrise; with the Athenians, Jews and various other ancient peoples and with certain Christian sects at sunset; with the Umbrians at noon or midday; and with the Roman and Egyptian priests at midnight.[14]

The next obvious division of time was the month; it was originally the length of time from one new-moon or one full-moon to the next. This was generally used by the ancient nations and served as

(14) Ancient Calendars and Cycles by Plumett.

the greater unit for many thousands of years. However, as science made considerable progress, it became apparent, as in the case of the day, that there are several kinds of months. There is a sidereal month, the time required for a passage of the moon about the earth as observed with reference to the fixed stars, namely, 27 days 7 hours 43 minutes 11·5 sec. There is also the synodic month, the interval of time from one conjunction of the sun and the moon to the next one, the average length of which is 29 days 12 hours 44 min. 3 sec. or two days 5 hours 0 min. 51·5 sec. more than the sidereal month. This is the month used by those among whom a lunar calendar is in vogue. It is the basis of the artificial month twelve of which make our common year.

Less obvious than the day or the month is the year, both sidereal and tropical. A sidereal year is the time taken by the sun to return to the same position relative to the fixed stars and a tropical year is the exact interval between two successive vernal equinoxes. But the ordinary civil year which is in use, contains an exact number of days, viz. 365 and the time taken by the sun to complete a revolution in the ecliptic is about 365¼ days. Probably at first the year recognised by the primitive nations was the lunar year consisting of twelve synodical months and is even now used in some parts of the world, specially by the Mahammadans.

The least obvious division of time was the week. It seems very likely that it arose from the need for a longer period than the day and a shorter than the month. The Chaldeans were the first who named the seven days of the week after the respective presiding planets and it is believed that the Hindus got this practice of naming the seven days from the Chaldeans.[15] The Romans also named the days of the week after the supposed presiding planets. This division of the week into seven days is in vogue throughout the world. But recently in Sōviet Russia there has been an attempt to make the week contain five days, to be named after the five principal planets excluding the sun and the moon. Probably the week was conceived to be made up of seven days as according to the ancient nations there were seven planets including the sun and the moon.

In all the ancient calendars there were usually twelve hours in the day and twelve hours in the night. It is still a mystery why the number twelve was chosen. Some suggest that the reason is found in the custom relating to the Babylonian knowledge of the inscribed hexagon. Some again say that this was selected because from twelve

(15) Indian Astronomy (in Marathi language) by S. B. Dikshit p. 138 f.

the fractions ½, ⅓, ¼ could easily be obtained. The day hours are longer than the night hours in the summer, and shorter in the winter. This fact is referred to by several ancient writers.[16]

To find the hours of the day the shadow cast by some obstruction to the sun's rays was used. Then an artificial gnomon was erected and lines were drawn on the earth to mark off the shadows. Since the hour-shadow is longer when the sun is near the horizon, either concave surfaces or curved lines on a plane were placed at the foot of the gnomon. The sundial was first used in Babylonia, in India and in Egypt. From Babylonia it came to Greece where it was introduced by Anaximander (c. 575 B.C.). In the sundial of that period the gnomon was placed at the centre of three concentric circles. But very often there were difficulties with the gnomon. At times when the sky was cloudy the gnomon did not cast a distinct shadow. Hence the need was early felt for some kind of device to tell the hours at night as well as during the day in cloudy weather and also when the sun's rays were direct. Various methods were employed by the early nations, such as burning of tapers, hour-glasses and water-clocks (or clepsydra). The hour-glass was known as early as 250 B.C. Plato gave much thought to the subject, and his conclusions might have suggested to Ctesibius (c. 150 B.C.) the idea of a water-clock. The Clepsydra was of earlier origin. There is mention of this instrument in the Vedāngo Jhotiṣa (500 B.C.). The Clepsydra was introduced into Rome in 159 B.C. in the early form water trickled from one receptacle to another in a given time, as the sand in an hour-glass. It is till recently in operation in one of the ancient towers of Canton in China. All these instruments led to the invention of the clock. In the middle ages Boethius (480 A.D. to 525 A.D.) invented a clock (c. 510 A.D.) and it was used in churches as early as 612 A.D. The invention of clocks driven by weights is ascribed to Pacificus, archdeacon of Verona, in the 9th century A.D. The pendulum clock was introduced about 1657 A.D. and was chiefly due to Huygens.

An attempt for the measurement of time and to fix the hour of the day as well as to prepare a calendar for practical purposes was as ancient as the human race. The evolution has, no doubt, been gradual, but the introduction of time in chronology dates to a prehistoric period and it has passed through various stages from the ancient down to the modern age.

(16) History of Mathematics, Vol. II, pages 668-670, by D. E. Smith.

SOME PROPERTIES OF BINOMIAL COEFFICIENTS

BY

P. KESAVA MENON, *Madras Christian College*.

§ 1·1. The following properties of binomial coefficients are easily proved by operational methods.* We shall use the symbol E defined by $E f(n) = f(n+1)$ which obeys the fundamental laws of algebra and for which, therefore, the expansions and identities hold good.

§ 1·2. Let $F(n)$ denote the expression

$$f(0) + \binom{n}{1} f(1) + \binom{n}{2} f(2) + \cdots + \binom{n}{t} f(t) + \cdots + \binom{n}{n} f(n).\dagger$$

Then it can be shown that

$$\frac{f(0)}{r} + \frac{nf(1)}{r(r+1)} + \frac{n(n-1)}{r(r+1)(r+2)} f(2) + \cdots + \frac{n!\ f(n)}{r(r+1)\cdots(r+n)} =$$

$$\frac{F(0)}{n+r} + \frac{nF(1)}{(n+r)(n+r-1)} + \frac{n(n-1)\ F(2)}{(n+r)(n+r-1)(n+r-2)} + \cdots + \frac{n!\ F(n)}{(n+r)(n+r-1)\cdots r} \quad \ldots \quad (1)$$

If r be an integer, by expanding $\dfrac{(r-1)!\ n!}{(n+r)!} \dfrac{(1+t)^{n+r}}{1-tE} f(0)$ in two ways and equating the coefficients of t^n we get formula (1). If we multiply throughout by $r(r+1) \cdots (r+n)$, both sides become expressions of the $(n+1)$th degree in r. Therefore, as the two sides are equal for all positive integral values of r, (1) is in identity. [The same reasoning may be employed for the following results also].

§ 1·3. Let $f(n)$ be x^n. Then $F(n)$ is equal to $(1+x)^n$.

And so, by (1)

$$\frac{1}{r} + \frac{nx}{r(r+1)} + \frac{n(n-1)x^2}{r(r+1)(r+2)} + \cdots + \frac{n!\ x^n}{r(r+1)\cdots(r+n)} =$$

$$\frac{1}{n+r} + \frac{n(1+x)}{(n+r)(n+r-1)} + \frac{n(n-1)(1+x)^2}{(n+r)(n+r-1)(n+r-2)} + \cdots + \frac{n!\ (1+x)^n}{(n+r)\cdots r}. \quad \ldots \quad (2)$$

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* Proofs by induction can be given but they are long and tedious.

† $\binom{n}{r}$ denotes the binomial coefficient nCr.

Differentiating both sides p times and equating coefficients of x^{q-p} we get

$$\frac{(n,q)\,(q,p)}{r\,(r+1)\cdots(r+q)} = \frac{(n,q)\,(q,p)}{(n+r,\,q+1)} + \frac{(n,\,q+1)(q+1,\,p)}{(n+r,\,q+2)}\cdot\binom{q-p+1}{q-p} + \cdots$$
$$+ \frac{(n,\,q+s)(q+s,\,p)}{(n+r,\,q+s+1)}\cdot\binom{q-p+s}{q-p} + \cdots$$
$$+ \frac{n!\,(n,p)}{(n+r,\,n+1)}\cdot\binom{n-p}{q-p},$$

where $(a,\,p)$ denotes $a\,(a-1)\cdots(a-p+1)$. Changing n into $n+q$ and reducing we get

$$(n+q+r,\,n)\cdot q\,! = (r,\,n)\cdot q\,! + \binom{n}{1}(r,\,n-1)\cdot(q+1)\,! + \cdots$$
$$+ \binom{n}{s}(r,\,n-s)\cdot(q+s)\,! + \cdots + (q+n)\,! \qquad \ldots \quad (3)$$

§ 2. $$\frac{(p+q)\,!}{q\,!\;x\,(x+1)\cdots(x+p+q)} = \frac{1}{x\,(x+1)\cdots(x+q)}$$
$$-\frac{\binom{p}{1}}{(x+1)\cdots(x+q+1)} + \cdots + (-1)^r\frac{\binom{p}{r}}{(x+r)(x+r+1)\cdots(x+r+q)}$$
$$+ \cdots + (-1)^p\frac{1}{(x+p)\cdots(x+p+q)} \qquad \ldots \quad (4)$$

For, if we denote $\dfrac{(-1)^s}{x+s}$ by $f(s)$, and $\dfrac{p\,!}{x\,(x+1)\cdots(x+p)}$ by $\mathrm{F}\,(p)$,

$$\mathrm{F}(p+q) = \frac{(p+q)\,!}{x\,(x+1)\cdots(x+p+q)} = (1+\mathrm{E})^{p+q}f(0).$$

The right side, multiplied by $q\,!$, is equal to

$$(1+\mathrm{E})^q f(0) + \binom{p}{1}(1+\mathrm{E})^q f(1) + \cdots + \binom{p}{p}(1+\mathrm{E})^q f(p)$$
$$= (1+\mathrm{E})^q\,(1+\mathrm{E})^p f(0).$$

Thus we get the required result.

§ 3. If $\phi(n)$ denotes $f(0) + \binom{n}{1}f(1) + \binom{n+1}{2}f(2) + \binom{n+2}{3}f(3) + \cdots$,

then

$$\frac{\phi(1)}{n+r} + \frac{n\phi(2)}{(n+r)\,(n+r-1)} + \frac{n(n-1)\,\phi(3)}{(n+r)\,(n+r-1)\,(n+r-2)} + \cdots + \frac{n\,!\;\phi(n+1)}{(n+r)\cdots r}$$
$$= \frac{f(0)}{r} + \frac{n+r+1}{r(r+1)}f(1) + \frac{(n+r+1)\,(n+r+2)}{r\,(r+1)\,(r+2)}f(2) + \cdots\cdots \qquad \ldots \quad (5)$$

By expanding $\dfrac{n!\,(r-1)!}{(n+r)!} \quad \dfrac{(1+t)^{n+r}}{1-\mathrm{E}\,(1+t)}\, f(0)$ in two ways and equating coefficients of t^n we get the required result.

§ 4. With the same notation as in § 3,

$$\frac{\phi(n+1)}{f(0)} = \frac{\dfrac{f(0)}{r} + \dfrac{n+r+1}{r(r+1)}\, f(1) + \dfrac{(n+r+1)(n+r+2)}{r\,(r+1)\,(r+2)}\, f(2) + \cdots\cdots}{\dfrac{f(0)}{r} - \dbinom{n}{1}\dfrac{f(1)}{r+1} + \dbinom{n}{2}\dfrac{f(2)}{r+2} - \cdots\cdots} \qquad \ldots \quad (6)$$

It can readily be shown that

$$(1-\mathrm{E})^{n+1}\left\{\frac{1}{r} - \binom{n}{1}\frac{\mathrm{E}}{r+1} + \binom{n}{2}\frac{\mathrm{E}^2}{r+2} - \cdots\cdots\right\} =$$

$$\frac{1}{r} + \frac{n+r+1}{r(r+1)}\,\mathrm{E} + \frac{(n+r+1)(n+r+2)}{r\,(r+1)\,(r+2)}\,\mathrm{E}^2 + \cdots\cdots$$

Operating on $f(0)$ by both sides of the above we get the required result.

§ 5. If $\phi\,(r,s) = f(s) + r\,f(s+1) + \dfrac{r\,(r-1)}{2\,!}\, f(s+2) + \cdots\cdots\cdots$ then

$$\frac{\phi\,(n,0)}{2n+1} + \frac{n\cdot\phi\,(n-1,0)}{(2n+1)\cdot 2n} + \frac{n(n-1)\,\phi\,(n-2,0)}{(2n+1)\cdot 2n\cdot(2n-1)} + \cdots + \frac{n!\,\phi\,(0,0)}{(2n+1)\cdot 2n\cdots(n+1)}$$

$$= \frac{\phi\,(n+1,0)}{n+1} + \frac{2n+2}{(n+1)\,(n+2)}\,\phi\,(n+1,1)$$

$$+ \frac{(2n+2)\,(2n+3)}{(n+1)\,(n+2)\,(n+3)}\,\phi\,(n+1,2) + \cdots\cdots \qquad \ldots \quad (7)$$

This may be proved by expanding $\dfrac{(n\,!)^2}{(2n+1)\,!} \quad \dfrac{(1+t)^{2n+1}\,(1+\mathrm{E})^{n+1} f(0)}{1-\mathrm{E}\,(1+t)}$ in two ways and equating the coefficients of t^n.

§ 6. If $\psi\,(n) = f(0) + f(1) + \cdots + f(n)$, then

$$(r+n,n)\,f(0) + (n,1)(r+n-1,n-1)\,f(1) + \cdots + (n,s)\,(r+n-s,n-s)\,f(s) + \cdots$$

$$= (r+n-1,n)\,\psi\,(0) + (n,1)\,(r+n-2,n-1)\,\psi\,(1) + \cdots\cdots$$

$$+ (n,s)(r+n-s-1,n-s)\,\psi\,(s) + \cdots\cdots \qquad \ldots \quad (8)$$

where (a, p) denotes, as in § 1 · 3, $a(a-1)\cdots\cdots(a-p+1)$.

Since $\psi\,(n) = \dfrac{1-\mathrm{E}^{n+1}}{1-\mathrm{E}}\, f(0)$, the right side of (8)

$= n\,! \times$ coefficient of t^n in

$$(1+t)^{n+r-1}\left\{1 + \frac{t}{1+t}\cdot\frac{1-\mathrm{E}^2}{1-\mathrm{E}} + \frac{t^2}{(1+t)^2}\cdot\frac{1-\mathrm{E}^3}{1-\mathrm{E}} + \cdots\cdots\right\} f(0)$$

$= n! \times$ coefficient of t^n in

$$\frac{(1+t)^{n+r}}{t}\left\{\frac{t}{1+t}(1-\mathrm{E}) + \frac{t^2}{(1+t)^2}(1-\mathrm{E}^2) + \cdots\right\}\frac{1}{1-\mathrm{E}}$$

$= n! \times$ coefficient of t^n in

$$\frac{(1+t)^{n+r}}{t}\left\{\frac{1}{1-\frac{t}{1+t}} - \frac{1}{1-\frac{t\,\mathrm{E}}{1+t}}\right\}\frac{1}{1-\mathrm{E}}$$

$$= n! \times \text{coefficient of } t^n \text{ in } \frac{(1+t)^{n+r+1}}{t}\left\{\frac{1+t(1-\mathrm{E})-1}{(1+t-t\,\mathrm{E})(1-\mathrm{E})}\right\}$$

$$= \text{coefficient of } \quad (1+t)^{n+r}\left\{1-\frac{t\,\mathrm{E}}{1+t}\right\}^{-1}$$

$= n! \times$ coefficient of t^n in

$$\{(1+t)^{n+r} + t(1+t)^{n+r-1}\,\mathrm{E} + t^2(1+t)^{n+r-2}\,\mathrm{E}^2 + \cdots\} f(0)$$

= the left side.

§ 6·2. If $f(r)$ is taken to be $\dfrac{x(x+1)\cdots(x+r-1)}{r!}$ and $f(0)$ to be 1, in (8), we have

$$\mathrm{F}(n) = 1 + \frac{x}{1!} + \frac{x(x+1)}{2!} + \cdots + \frac{x(x+1)\cdots(x+n-1)}{n!}$$

$$= \frac{(x+1)(x+2)\cdots(x+n)}{n!},$$

and so

$$(r+n,\,n) + \binom{n}{1}(r+n-1,\,n-1)x + \binom{n}{2}(r+n-2,\,n-2)x(x+1) + \cdots$$

$$= (r+n-1,\,n) + \binom{n}{1}(r+n-2,\,n-1)(x+1)$$

$$+ \binom{n}{2}(r+n-3,\,n-2)(x+1)(x+2) + \cdots \qquad \ldots \quad (9)$$

My thanks are due to Dr. V. Ganapathy Iyer, for his valuable suggestions.

ON THE THREE CONICOIDS OF A TANGENTIAL SYSTEM WHICH PASS THROUGH A POINT

BY

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1. If $\sum = 0$ and $\sum' = 0$ be the tangential equations of two given conicoids, $\sum + \lambda \sum' = 0$ represents a conicoid touching the common tangent planes to the two conicoids i.e., inscribed in the developable common to the two conicoids. Transforming to point coordinates, it will be seen that the equation is a cubic in λ and thus, in general, there will pass through any point in space three members of the system. The discriminant of the above cubic equated to zero gives a surface of the eighth degree on one side of which—we shall call this side the inside—the three roots are all real and on the other side only one root is real. The object of the present paper is to discuss the nature of the three conicoids, when real, through a point.

2. If the common self-conjugate tetrahedron of the two conicoids be taken as the reference tetrahedron, the equation to the system of conicoids in quadriplanar coordinates will be $\sum \dfrac{x^2}{a - \lambda a'} = 0$ where $\sum x^2/a = 0$ and $\sum x^2/a' = 0$ are the equations to the two given conicoids. A conicoid of the system is a hyperboloid of one sheet if two of the coefficients a, b, c, d are positive and the other two negative whereas for an ellipsoid or a two-sheeted hyperboloid, three of the coefficients are of the same sign (say positive) and the fourth negative. To distinguish now between an ellipsoid and a two-sheeted hyperboloid, we consider the section by the plane at infinity which is real for a hyperboloid and imaginary for an ellipsoid. The equation $\sum x^2/a = 0$ with a, b, $c > 0$ and $d < 0$ will represent an ellipsoid if $a + b + c + d < 0$ and a hyperboloid of two sheets if $a + b + c + d > 0$. The condition for a paraboloid is plainly $a + b + c + d = 0$.

3. In what follows two conicoids of the system will be called like when they are conicoids of the same kind with corresponding coefficients of like sign. Clearly we can start with two like conicoids. For the signs of $a - \lambda a'$, $b - \lambda b'$ etc. are the same as, or opposite to those of a', b' etc. if λ is less than the least or greater

than the greatest of a/a', b/b', c/c', d/d'. And as

$$\sum a - \lambda \sum a' = \sum a'(\sum a/\sum a' - \lambda),$$

we take $\lambda < \sum a/\sum a' = \lambda_0$ if λ_0 is less than the least of a/a', b/b', c/c', d/d' or $\lambda >$ the greatest of λ_0, a/a', b/b', c/c', d/d' if λ_0 is greater than one or more of a/a', b/b', c/c', d/d'.

We suppose that the two given conicoids are not both paraboloids for in that case all the members of the system will also be paraboloids. It will be easily seen that the two conicoids to start with can always be taken to be two like ellipsoids except in one case only. If S and S′ are two like hyperboloids of one sheet we choose λ_1, λ_2 greater than three and less than the fourth of the ratios a/a', b/b', c/c', d/d' so that the equations $\sum x^2/(a-\lambda_1 a')=0$ and $\sum x^9/(a-\lambda_2 a')=0$ will represent both like ellipsoids or hyperboloids of two sheets. Taking therefore S and S′ to be like hyperboloids of two sheets with a, b, c; a', b', $c'>0$ and d, $d'<0$ we may suppose d/d' to be smaller than all or greater than only one of a/a', b/b', c/c' (otherwise we take $\lambda = 1/\mu$). If d/d' were the smallest, $\lambda_0 < d/d'$ and we choose λ_1, λ_2 so as to lie between d/d' and λ_0. If $a/a' > b/b' > d/d' > c/c'$ there is no upper or lower bound for λ_0. In this case we can easily get λ_1, λ_2 so that $\Sigma x^2/(a-\lambda_1 a')=0$ and $\Sigma x^2/(a-\lambda_2 a')c'=0$ represent like ellipsoids except when S and S′ are hyperboloids of two sheets and

$$a/a' > \lambda_0 > b/b' > d/d' > c/c'.$$

4. Let then $S \equiv \Sigma x^2/a = 0$ and $S' \equiv \Sigma x^2/a' = 0$ be two like ellipsoids. Let us first suppose $a/a' > b/b' > c/c' > d/d'$ where a, b, c; a', b', $c' > 0$ and d, $d'<0$. Then $d/d' > \lambda_0$. The signs of $f(\lambda) \equiv \Sigma(b-\lambda b')(c-\lambda c')(d-\lambda d')x^2$ for values $-\infty$, d/d', c/c', b/b', a/a', $+\infty$ of λ are those of $-S'$, $+$, $+$, $-$, $+$, $+S'$. The roots of $f(\lambda)=0$ are therefore all real and the developable is imaginary in this case. One root lies in $(c/c', b/b')$ and another in $(b/b', a/a')$. The third root lies in $(-\infty, d/d')$ or $(a/a', +\infty)$ according as $S' \gtrless 0$ i.e., according as the point lies outside or inside S′. The root which lies in $(c/c', b/b')$ determines a two-sheeted hyperboloid and that in $(b/b', a/a')$ a hyperboloid of one sheet. Inside S′, the third root gives an ellipsoid. Outside S′, it lies in $(d/d', -\infty)$. It therefore gives an ellipsoid or a two-sheeted hyperboloid according as $\lambda \lessgtr \lambda_0$ i.e., according as $f(\lambda_0) \gtrless 0$. Now $f(\lambda_0)=0$ is the equation of the unique paraboloid P of the system and thus in this case "The three conicoids through a point are always real one a hyperboloid of one sheet, the other of two sheets and the third an ellipsoid or a two-sheeted hyperboloid

according as the point lies inside or outside the unique paraboloid P of the system."

5. Let us next suppose $a/a' > b/b' > d/d' > c/c'$. The signs of $f(\lambda)$ for values $-\infty$, c/c', d/d', b/b', a/a', ∞ of λ are those of $-S'$, $-$, $-$, $-$, $+$, S'. Within S' the roots are real lying in $(-\infty, c/c')$, $(b/b', a/a')$, $(a/a', \infty)$. The middle root which is always real gives a one-sheeted hyperboloid. If $\lambda_0 < c/c'$, the paraboloid $f(\lambda_0)=0$ is elliptic and within it $(f(\lambda_0)>0)$ the three conicoids are an ellipsoid, an hyperboloid of one sheet and one of two sheets. If $\lambda_0 > a/a'$, the conicoids are again an ellipsoid, an hyperboloid of two and one sheets for points within the paraboloid $f(\lambda_0)=0$* which is again an elliptic paraboloid.

If λ_0 lies in $(c/c', a/a')$ the two extreme roots for $S'<0$ are both ellipsoids and this region corresponds to $f(\lambda_0)<0$ i.e., outside the paraboloid $f(\lambda_0)=0$. Outside, the signs are $-$, $-$, $-$, $-$, $+$, $+$. Inside P $(f(\lambda_0)>0)$ the other two roots are real if λ_0 does not lie between b/b', a/a'. Supposing $c/c'<\lambda_0<d/d'$, the paraboloid P is hyperbolic and within it the two roots give both hyperboloids of one sheet, whereas if $d/d'<\lambda_0<b/b'$, the two roots for points within P give one an ellipsoid and the other a two-sheeted hyperboloid. The paraboloid P is elliptic in this case.

We have yet to examine the region outside S' and outside P i.e. $S'>0$ and $f(\lambda_0)<0$ (except when $\lambda>a/a'$). Writing $\lambda=\mu+\lambda_0$ we get

$$f(\lambda)\equiv f(\mu+\lambda_0)\equiv(-a'b'c'd')\mu^3+\mu^2 T+\mu T+f(\lambda_0)=0$$

has always one root greater than λ_0 ($<\lambda_0$ if $\lambda_0>a/a'$). Hence the other two roots whenever real are both less or both greater than λ_0. Hence if $T>0$ the two roots when real are both $<\lambda_0$ and if $T<0$ and $T'<0$ both are again $<\lambda_0$. But the nature of the corresponding conicoids cannot be determined unless the particular interval in which the roots lie is found out.

6. There is one case yet to be considered viz. when the conicoids S and S' are both like ellipsoids (or 2 sheeted hyperboloids) and $a/a'>\lambda_0>b/b'>d/d'>c/c'$. The signs of $f(\lambda)$ for $-\infty$, c/c', d/d', $b'b'$, a/a', ∞ are those of $-S'$, $-$, $-$, $-$, $+$, $+S'$. One root is always real, lies in $(b/b', a/a')$ and determines a hyperboloid of one sheet. Within S', the other two roots are real and determine two ellipsoids (or 2 sheeted hyperboloids). Within the developable the other two roots are real and they are both less or both greater than λ_0. But

* As three of $a-\lambda a'$, $b-\lambda b'$ etc. are negative.

as in the preceding section unless the particular interval in which the roots lie is determined it is not possible to determine the nature of the two conicoids. All that can be said is that they are both ellipsoids or both hyperboloids of one or two sheets. Hence summing up we have "*of the three conicoids through a point one is always real and is a hyperboloid of one sheet. Within the unique paraboloid P of the system—whenever it is elliptic, the other two conicoids are an ellipsoid and a two-sheeted hyperboloid, whereas if P is hyperbolic the three conicoids are all one sheeted hyperboloid or one a hyperboloid of one sheet and the other two both ellipsoids or two sheeted hyperboloids. Outside P the two conicoids whenever real are both ellipsoids, or hyperboloids of one or two sheets.*" The case when two or more of the ratios were equal or λ_0 equal to one of these could be similarly treated.

7. If the two conicoids to start with are both paraboloids we may take both like elliptic paraboloids. If $a/a' > b/b' > c/c' > d/d'$ the three parabolids through a point are always real and two of them elliptic, the third hyperbolic. If however $a/a' > b/b' > d/d' > c/c'$ one of them is always real and is a hyperbolic paraboloid. The other two, whenever real, are both elliptic or both hyperbolic.

METHODS OF CONSTRUCTING MAGIC SQUARES

BY

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I

A method for constructing Magic Squares of the Concentric type having an odd number of cells in each row.

1. We gave a method for constructing Magic Squares of the concentric type, having an even number of cells in each row, in Vol. IV of this journal. We give below a method applicable to similar magic squares, having an odd number of cells in each row. The symbols K_n and S_n and terms such as " n - cell-square," complement of a number " etc., have the same meaning here as in the earlier paper.

2. The method we give below consists of first filling in numbers in the outermost belt, then in the one next to it and so on until we come to a 3 - cell square, "the Basic Magic Square." The same method, with slight modifications is used to fill up every one of the belts. But this method will not be applicable to the Basic Magic Square, which, however, may be constructed by the usual method of constructing a magic square of an odd order.

3. To Fill up the outermost belt of a $(2n+1)$-cell-square.

The outermost belt of a $(2n+1)$ - cell-square consists of $(2n+1)^2-(2n-1)^2$ or 8 n cells. These cells have to be filled up by the first $4n$ natural numbers and their complements (got by subtracting from K_{2n+1}). This leaves the $(2n-1)^2$ numbers commencing from $4n+1$ and ending with $4n^2+1$ to be filled in the $(2n-1)$ - cell-square which is inside the belt.

Now let us write the numbers 1 to $4n$ in two rows so that there are an equal number of them in each row, as given below :—

A	$1, 2, 3, 4, \cdots\cdots, n,$	$n+1, n+2, \cdots\cdots, 2n,$	B
C	$2n+1, 2n+2, \cdots, 3n,$	$3n+1, \cdots\cdots, 4n-1, 4n.$	D

* The author is very much indebted to his friend Vidwan K. Lakshminarayana Srowthy of Sri Chamarajendra Vedamahapatasala, Bangalore for much help in the preparation of the paper and the testing of the method.

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They are to be divided into four equal groups A, B, C and D as shown above. The first number in group C i.e. $2n+1$ is placed in the topmost cell of the right column of the belt and the second number from the end in group D i.e. $4n-1$ is placed in the bottommost cell of the same column. All the remaining numbers in group C excepting the last, along with the second number in group B i.e. $n+2$, are placed anywhere in this column. The numbers in group A are put in the left column of the belt so that they become the beginning numbers of rows in which there is no number in the end.

Now we come to the top and bottom rows. All the numbers in group B, excepting the second number which has already been placed elsewhere, may be placed in any of the cells of the top row, excepting the first cell. The first cell of the bottom row is not to be filled up at present and the last cell is, already, filled up. The last number in group C i.e. $3n$ and all the numbers of group D, excepting the one which is already placed elsewhere, are placed in the n cells of the bottom row, which are exactly below the unoccupied cells of the top row.

Now, we have filled up half of the cells in this belt. The other half is to be filled up with the complements of the numbers already put in. The complements † of $2n+1$ and $4n-1$ are put in the other end of the diagonals in which they are situated. The complements of other numbers are put in the other end of the row or column in which they are placed. Now the outermost belt is filled up.

The sum of the numbers first placed (i.e. excluding the complements of the numbers, introduced later on) in the right column is

$$(2n+1)+(2n+2)+\cdots+(3n-1)+(4n-1)+(n+2)=\frac{5n(n-1)}{2}+5n+1$$
$$=(5n^2+5n+2)/2.$$

The sum of the numbers first placed in the left column is

$$1+2+\cdots+n=(n^2+n)/2.$$

The sum of the numbers first put in the top row is

$$(n+1)+(n+3)+(n+4)+\cdots+2n+(2n+1)=n+1+\frac{(3n+4)(n-1)}{2}$$
$$=(3n^2+3n-2)/2\ ;\ \text{and}$$

the sum of the numbers first placed in the bottom row is

$$3n+(3n+1)+\cdots+(4n-2)+(4n-1)+4n=\frac{7n(n+1)}{2}=\frac{7n^2+7n}{2}.$$

† As we are dealing with numbers in a $(2n+1)-$cell square the complement of any number is got by subtracting it from K_{2n+1} i.e. $(2n+1)^2+1$.

Hence the sum of all the numbers in the right column is equal to $\frac{1}{2}(5n^2+5n+2)$ + sum of the Complements of the n numbers in the left column,

$$= \frac{5n^2+5n+2}{2} + \left[nK_{2n+1} - \frac{n^2+n}{2}\right] = nK_{2n+1} + (2n^2+2n+1)$$
$$= \left(n+\frac{1}{2}\right)K_{2n+1} = S_{2n+1}.$$

Similarly the sum of the numbers in the left column is

$$\frac{n^2+n}{2} + \left[(n+1)K_{2n+1} - \frac{5n^2+5n+2}{2}\right] = (n+1)K_{2n+1} - (2n^2+2n+1)$$
$$= \left(n+\frac{1}{2}\right)K_{2n+1} = S_{2n+1}.$$

The sum of the numbers in the top row is

$$\frac{3n^2+3n-2}{2} + \left[(n+1)K_{2n+1} - \frac{7n^2+7n}{2}\right] = (n+1)\,K_{2n+1} - (2n^2+2n+1)$$
$$= \left(n+\frac{1}{2}\right)K_{2n+1} = S_{2n+1};$$

and the sum of the numbers in the bottom row is

$$\frac{7n^2+7n}{2} + \left[nK_{2n+1} - \frac{3n^2+3n-2}{2}\right] = nK_{2n+1} + (2n^2+2n+1) = \left(n+\frac{1}{2}\right)K_{2n+1}$$
$$= S_{2n+1}.$$

Thus the sum of the numbers in each of the two rows and columns of the belt is S_{2n+1} as it should be. In each of the columns except the right and left columns and in each of the rows other than the top and bottom rows, as well as in each of the diagonals we have placed only two numbers whose sum is K_{2n+1}.

4. **To fill up the cells in the remaining belts.**

The above method is used to fill up the remaining belts also, until we come to the Basic Magic square. But as we have already used the first $4n$ numbers and their complements in the first ‡ belt we begin with $4n+1$ instead of with 1 in filling up the second belt. This belt will be the same as the outermost belt of a $(2n-1)$-cell-square, every number in which is increased by $4n$. When finding the complement of a number, the number should always be subtracted from K_{2n+1}, irrespective of the belt in which the number is placed.

After thus filling up all the belts, we fill up the basic magic square with the numbers $2n^2+2n-3,\ 2n^2+2n-2,\ \cdots,\ 2n^2+2n+5$.

5. We can easily prove that the diagonals as well as the rows and columns, not taken into consideration in § 4 above, also give the total S_{2n+1}.

‡ In the earlier paper the belts were numbered from inside, but here they are numbered from the outside.

If we remove the outermost belt of a $(2n+1)$ - cell - square constructed by the above method, we get a $(2n-1)$ - cell - square and the linear total of this square will e $\frac{(2n-1)\,S_{2n+1}}{(2n+1)}$. In general if we remove the $(n-\gamma)$ outermost belts of a $(2n+1)$ - cell - square, the resulting $(2\gamma+1)$ - cell magic square will have $\frac{(2\gamma+1)\,S_{2n+1}}{(2n+1)}$ for its linear total and so, if we subtract $\left(\frac{2\gamma+1}{2n+1}\,S_{2n+1} - S_{2\gamma+1}\right) \div (2\gamma+1)$ i.e. $2[(n^2+n)-(\gamma^2+\gamma)]$ from each of the numbers of this new magic square it will become a $(2\gamma+1)$ - cell magic square containing the natural numbers 1 to $(4\gamma^2+4\gamma+1)$.

6. A 11 - cell - square, constructed by the above method is given below as an illustration.

The Outermost or First belt

A	1, 2, 3, 4, 5,	6, 7, 8, 9, 10,	B
C	11, 12, 13, 14, 15,	16, 17, 18, 19, 20.	D

11 (the first no. in group C) and 19 (the second number from the end in D) are placed in the topmost and bottom-most cells respectively of right column. 12, 13, 14 (the remaining nos. of C, excepting the last) and 7 (the second number in B) are placed in right column.

1, 2, 3, 4, 5 (the nos. in A) are placed in left column. 6, 8, 9, 10 (the remaining numbers in B) are placed in the top row.

15 (last no. in C) and 16, 17, 18, 20 (the remaining numbers of D) are placed in the bottom row.

The complements are got by subtraction from $K_{11} = 11^2 + 1 = 122$.

The Second belt

A	21, 22, 23, 24,	25, 26, 27, 28,	B
C	29, 30, 31, 32,	33, 34, 35, 36.	D

29 (first no. in C) and 35 (the second no. from the end in D) are placed in the top and bottom cells of right column. 30, 31 (the remaining nos. of C, excepting the last no.) and 26 (the second no. in B) are put in the right column. 21, 22, 23, 24 (nos. in A) are placed in the left column. 25, 27, 28 (the remaining nos. in B are placed in the top row. 32 (last no. in C), 33, 34, 36 (the remaining nos. in D) are placed in the bottom row.

The Third belt

A	37, 38, 39.	40, 41, 42,	B
C	43, 44, 45,	46, 47, 48.	D

43 (first no. in C) and 47 (second no. from end in D) are placed in the top and bottom cells of right column. 44 (the remaining no.

in C after excluding the last no.) and 41 (second no. in B) are put in the right column. 37, 38, 39 (nos. in A) are placed in left column. 40, 42 (the remaining nos. in B) are placed in top row. 45 (last no. in C), 46, 48 (the remaining nos. in D) are placed in the bottom row.

The Fourth belt

A	49, 50,	51, 52.	B
C	53, 54,	55, 56.	D

B } 53 (first no. in C) and 55 (second no.
D } from end in D) are placed in top right and bottom right corner cells. As no number but the last one remains in C, 52 (second no. in B) is put into a cell in the right column. 49 and 50 (nos. in A) are placed in left column. 51 (the remaining no. in B) is placed in top row. 54 (last no. in C) and 56 (the remaining no. in D) are placed in the bottom row.

The other cells in the belts are filled up with the complements of the above numbers and the § basic magic square with the numbers 57-65. The resulting magic square is reproduced below :—

103	6	8	9	10	107	106	105	104	102	11
110	87	90	89	88	86	25	27	28	29	12
109	92	75	77	76	74	40	42	43	30	13
108	91	78	67	68	66	51	53	44	31	14
115	96	81	70	64	57	62	52	41	26	7
5	21	37	49	59	61	63	73	85	101	117
4	22	38	50	60	65	58	72	84	100	118
3	23	39	69	54	56	71	55	83	99	119
2	24	79	45	46	48	82	80	47	98	120
1	93	32	33	34	36	97	95	94	35	121
111	116	114	113	112	15	16	17	18	20	19

II

A method of constructing 4n-cell-magic squares

1. In his book ‖ " Mathematical recreations and Essays ", Ball has given two methods of constructing magic squares of order $4n$.

§ The remaining 3-cell square is the basic magic square. See § 2 above.

‖ Fourth Edition 1905 (MacMillan & Co.) pages 149-152.

According to the first method, a subsidiary square is formed whose cells are filled up with the numbers 1 to 16 n^2, commencing with the first cell in the top row and proceeding from left to right, taking up the second row after the first and so on. The subsidiary square is magic in the diagonals. Ball has shown that this square can be made magic in rows and columns by interchanging $4n^2$ of the numbers in the subsidiary square with the numbers in the complementary ¶ cells. Thus $8n^2$ numbers change their places, while the rest remain where they were first placed. The $8n^2$ cells, the numbers in which are to be thus changed, must be so selected that exactly $2n$ of them are to be in every row or column.

Ball gives two methods of selecting these cells. Here we propose to give a ** third method.

2. Each of the diagonals of the square passes through 2 quadrants. The cells in these two diagonals, along with those cells which are diagonally related to them, being also situated in the quadrants through which the respective diagonals pass, form one group. Or if we imagine the cells in the square to be coloured black and white alternately as in a chess-board,†† the black cells in the first and third quadrants along with the white cells in the second and fourth quadrants form one group. The remaining cells form another group, i.e.,

1st, 3rd, 5th, . . . , $(2n-1)$th, $(2n+2)$th, $(2n+4)$th, etc . . . . $4n$th cells in the rows of the same no. (i.e. 1st, 3rd etc. rows) and 2nd, 4th, 6th, . . . , $2n$th, $(2n+1)$th, $(2n+3)$th, . . . , $(4n-1)$th cells in the remaining rows form one group and the rest another.

The numbers in the cells of one group are left unchanged but the numbers in the cells of the other group are replaced by their complements (got by subtracting from K_{4n}) or, in other words, by the numbers in the complementary cells. The resulting square is magic.

¶ A $4n$-cell square may be divided into 4 equal squares by drawing a thick line between the $2n$th and $(2n+1)$th columns as well as between the $2n$th and $(2n+1)$th rows. We may call these 4 squares by the name "Quadrants" and the points where the four quadrants meet may be called the origin (as in a graph). Any two cells which are similarly situated with reference to the dividing lines are called complementary cells; i.e., the pth cell in the qth row is complementary to the $(4n-p+1)$th cell in the $(4n-q+1)$th row. In the subsidiary square above, complementary cells contain numbers which are complements to each other.

** For a 4-cell square this method gives the same result as Ball's method.

†† In the squares black coloured cells may be denoted by small dots put in the cells. This makes selection of cells easy.

3. We may prove, as follows, that it is a magic square :

Any row contains, exactly, $2n$ pairs of horizontally-related ‡ cells. The sum of the numbers in any such pair of cells in the rth row is $4n(2r-1)+1$. n out of these $2n$ pairs belong to one of the groups (stated above) and the rest to the other. So the sum of the numbers in this row, which belong to each of the groups is same ; and, hence, the sum of their complements also must be the same. So after the interchanges are over, i.e. in the magic square, each row will contain $2n$ "original" numbers and $2n$ other numbers, whose sum is equal to the sum of the complements of the $2n$ "original" nos. in the row. So the sum of all the nos. in any row is $2n\,K_{4n}$ i.e. S_{4n} as it should be. Similarly, it can be proved that the sum of the numbers in any column is also S_{4n}.

4. A 8-cell magic square has been constructed by this method and given below as an illustration.

1st, 3rd, 6th and 8th cells in the first, third, sixth and eighth rows and the 2nd, 4th, 5th and 7th cells in the second, fourth, fifth and seventh rows form one group. The numbers in the cells of this group are unaltered but the numbers in the other cells have been replaced by their complements.

The numbers, which are unaltered, are printed in ordinary type and the numbers, which have replaced their complements in italics.

1	*63*	3	*61*	*60*	6	*58*	8
56	10	*54*	12	13	*51*	15	*49*
17	*47*	19	*45*	*44*	22	*42*	24
40	26	*38*	28	29	*35*	31	*33*
32	34	*30*	36	37	*27*	39	*25*
41	*23*	43	*21*	*20*	46	*18*	48
16	50	*14*	52	53	*11*	55	*9*
57	*7*	59	*5*	*4*	62	*2*	64

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‡ Two cells in the same row, which are at equal distances from the two ends i.e. the mth and $(4n-m+1)$th cells are said to be horizontally related.

NOTES AND DISCUSSIONS

Lecture Notes

I. *Eccentricity of a Hyperbola*

To clear the ambiguity in the values of the eccentricity of a hyperbola given by the general equation, we offer the following tip:

THEOREM: *If* $ax^2+2hxy+by^2+2gx+2fy+c=0$ *be a hyperbola referred to axes inclined at an angle* ω, *the eccentricity corresponding to the real foci is greater than, equal to, or less than* $\sqrt{2}$ *according as* $\Delta\ (a+b-2h\cos\omega)$ *is greater than, equal to, or less than 0, where* Δ *is the usual invariant.*

The proof is immediate, by invariants.

For, if the canonical form be $a'x^2+b'y^2+c'=0$, and e the eccentricity corresponding to the real focus, e^2 is given by

$$1-\frac{a'}{b'} \text{ or } 1-\frac{b'}{a'} \text{ according as } a'c' \text{ is negative or positive.}$$

Let $a'c'$ be negative; then $b'c'$ is positive.

$$\text{Then } e^2-2 \gtreqless 0$$

$$\text{according as } \frac{a'+b'}{b'} \lesseqgtr 0$$

$$\text{i.e. } \frac{a'+b'}{a'b'c'} \gtreqless 0$$

$$\text{i.e. } \frac{a+b-2h\cos\omega}{\Delta} \gtreqless 0 \quad \text{(by invariants)}$$

$$\text{i.e. } \Delta\ (a+b-2h\cos\omega) \gtreqless 0$$

The same result is reached also when $a'c'$ is positive.

Cor. 1. An oblique hyperbola (i.e. not rectangular) is situated in the obtuse or acute angles between the asymptotes according as $\Delta\ (a+b-2h\cos\omega)$ is positive or negative.

Cor. 2. If Δ, Δ_1 denote the usual invariants corresponding to a hyperbola and its conjugate whose equations differ only in the constant term, $\Delta+\Delta_1=0$.

II. *The Jacobian of three conics touching one another at the same point*

The usual description of the above Jacobian is that it consists of a line and a conic (*Vide* p. 438, Askwith's *Analytical Geometry*, 1918). But the Jacobian actually consists of three concurrent straight lines made up of the common tangent at the point of contact and the double lines of the involution determined by the three pairs of common chords of the conics.

Taking Cartesian rectangular axes, with $y=0$ and $x=0$ for the tangent and normal at the point of contact, we write the equations of the three conics :

$$S_r + 2f_r y \equiv a_r x^2 + 2h_r xy + b_r y^2 + 2f_r y = 0 \qquad (r=1,2,3)$$

and their pairs of common chords through the origin

$$C_i \equiv f_j S_k - f S_{kj} = 0 \qquad (i \neq j \neq k,\ i, j, k = 1, 2, 3)$$

are evidently in involution, since $f_1 c_1 + f_2 c_2 + f_3 c_3 = 0$.

It is easily verified that the Jacobian of two pairs of lines through the origin is the same as the double lines of the involution determined by them. Hence, the double lines of the involution determined by the pairs of common chords are given by the Jacobian

$$\frac{\partial(C_i,\ C\ )}{\partial(x,\ y)} = 0$$

$$\text{i.e.} \quad \sum f_1 \cdot \frac{\partial(S_2,\ S_3)}{\partial(x,\ y)} = 0.$$

But the Jacobian of three conics in question is

$$\begin{vmatrix} \frac{\partial S_1}{\partial x} & \frac{\partial S_1}{\partial y} + 2f_1 & 2f_1\ y \\ \frac{\partial S_2}{\partial x} & \frac{\partial S_2}{\partial y} + 2f_2 & 2f_2\ y \\ \frac{\partial S_3}{\partial x} & \frac{\partial S_3}{\partial y} + 2f_3 & 2f_3\ y \end{vmatrix} = 0$$

$$\text{i.e.} \quad y \sum f_1 \frac{\partial(S_2,\ S_3)}{\partial\,(x,\ y)} = 0.$$

which proves our result.

III. *Lightning Elimination*

Problem: Eliminate (m, n) between

$$\frac{am^2+bm+c}{a'm^2+b'm+c'} = \frac{an^2+bn+c}{a'n^2+b'n+c'},$$

and $a''m^2+b''m+c'' = a''n^2+b''n+c'' = 0$

Note that (m, n) are the roots of two equations:

$$a''x^2+b''x+c''=0$$

$$(a-\lambda a')\,x^2+\,(b-\lambda b')\,x+c-\lambda c'=0$$

Hence $\dfrac{a-\lambda a'}{a''} = \dfrac{b-\lambda b'}{b''} = \dfrac{c-\lambda c'}{c''} = \mu$ (say)

Eliminating λ, μ, we get $\begin{vmatrix} a & b & c \\ a' & b' & c' \\ a'' & b'' & c'' \end{vmatrix} = 0.$

Application: To find the bisectors of the angles between the line-pair $ax^2+2hxy+by^2=0$, the angle between the axes being 'ω'.

The bisectors are given by

$$\frac{(y-m_1x)^2}{1+m_1{}^2+2m_1\ \cos\ \omega} = \frac{(y-m_2x)^2}{1+m_2{}^2+2m_2\ \cos\ \omega}$$

where $\quad ax^2+2\ hxy+by^2 \equiv b\ (y-m_1x)\ (y-m_2\ x)$

so that $\quad a+2hm_1+bm_1{}^2=0=a+2hm_2+bm_2{}^2$

Eliminating m_1, m_2 as in the original problem

$$\begin{vmatrix} x^2 & -\ xy & y^2 \\ 1 & \cos\ \omega & 1 \\ b & h & a \end{vmatrix} = 0$$

i.e. $\ x^2(a\ \cos\ \omega-h)+xy\ (a-b)+y^2\ (h-b\ \cos\ \omega)=0.$

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Some Properties of Recurring Decimals

1. THEOREM: *If p, and q be primes, and the scale of notation R be prime to p and q, and if*

$$\frac{h}{p} = \cdot \dot{a}_1 a_2 \cdot \cdot \cdot \cdot \cdot \dot{a}_r$$

$$\frac{k}{q} = \cdot \dot{b}_1 b_2 \cdot \cdot \cdot \cdot \cdot \dot{b}_{rl}$$

then the period-number of $\frac{k}{q}$ *that is the number represented in scale R by* $b_1 b_2 \cdot \cdot \cdot b_{rl}$ *is divisible by p.*

Proof:— $a_1 a_2 \cdot \cdot \cdot a_r = (\mathrm{R}^r - 1)h/p$

$b_1 b_2 \cdot \cdot \cdot b_{rl} = (\mathrm{R}^{rl} - 1)k/q$

Now, $\mathrm{R}^r - 1$ is a multiple of $\mathrm{R}^r - 1$ which is a multiple of p. Also $(p, q) = 1$. Hence $\mathrm{R}^{rl} - 1$ is a multiple of p.

$\therefore$ The period-number of k/q is divisible by p.

Also, if $l = 1$, then applying the property to each divisor, we get that

the period-number of the first division is divisible by the second divisor and *vice versa.*

2. *If n is the highest power of p which divides* $10^l - 1$ *where* $(l, p) = 1$ *and r is a positive integer then* † $10^c \equiv 1 \pmod{p^{n+r}}$ *if* $c = lp^r$ *and the period-number of* $1/p^n$ *and of* $1/p^{n+r}$ *leave the same remainder, when divided by p.*

Proof:—Let $(10^l - 1)/p^n = \mathrm{M}$ where $(\mathrm{M}, p) = 1$.

Then $(10^c - 1) / p^{n+r} =$ an integer N if $(\mathrm{N}, p) = 1$.

Further,

$$\begin{aligned} \mathrm{N} = (10^{lt} - 1)/p^{n+r} &= \{(\mathrm{M}p^n + 1)^t - 1\} / p^{n+r} \\ &= \{\mathrm{M}p^{n+r} + kp^{n+r+1}\} / p^{n+r} \\ &= \mathrm{M} + kp \end{aligned}$$

when $t = p^r$ and k is an integer.

Hence $\mathrm{N} \equiv \mathrm{M} \pmod{p}$.

So, if $1/p^n = \cdot \dot{a}_1 a_2 \cdot \cdot \cdot \cdot \cdot \cdot \dot{a}_l$

and $1/p^{n+r} = \cdot \dot{b}_1 b_2 , \cdot \cdot \cdot \cdot \dot{b}_c$

† *Vide* Hardy and Wright. *An introduction to the Theory of Numbers* pp. 65, 66.

then $a_1 a_2 \cdot \cdot \cdot \cdot a_l \equiv b_1 b_2 \cdot \cdot \cdot \cdot \cdot b_c \pmod{p}$

Also, the most usual value of n is 1.

In all such cases, the period-number of $1/p$ and the period-number of $1/p^{r+1}$ differ by a multiple of p.

That is, the period-number of $p^{-x} \equiv$ period-number of $p^{-1} \pmod{p}$ for any integer x. The same principle holds essentially for any radix, which is prime to p.

§ 3. In this paragraph I shall explain a simple method of obtaining the decimal expansions of reciprocals of powers of a prime p. Let us take $p=13$ by way of illustration.

By Fermat's Theorem $10^{12} \equiv 1 \pmod{13}$. Hence the number of decimal places n in 1/13, i.e., the least number n such that $10^n \equiv 1$, (mod 13) is a factor of 12. Here $n=6$ and in fact $\frac{1}{13} = \cdot \dot{0}7692\dot{3}$.

The period number (P. N) of 1/13 is thus $r=076923$. Since it is an integer the 0 at the beginning makes no difference, but it is convenient to retain it.

The number of places in the period of $1/13^2$ is 13×6. We shall now obtain the corresponding P. N.

We first divide r by 13 giving 005917 which are the first 6 digits in $1/13^2$ and leaving a remainder 2. To get the next 6 places add 2×076923 to 005917 thus getting 159763. The remainder at this stage is 4. Continuing thus we have

	Remainder
0 0 5 9 1 7	2
1 5 3 8 4 6	
1 5 9 7 6 3	4
1 5 3 8 4 6	
3 1 3 6 0 9	6
1 5 3 8 4 6	
4 6 7 4 5 5	8
1 5 3 8 4 6	
6 2 1 3 0 1	10
1 5 3 8 4 6	
7 7 5 1 4 7	12
1 5 3 8 4 6	

	Remainder
9 2 8 9 9 3	$14 \equiv 1 \pmod{13}$
1 5 3 8 4 6	
1, 0 8 2 8 3 9	3
1 5 3 8 4 6	
1, 2 3 6 6 8 5	5
1 5 3 8 4 6	
1, 3 9 0 5 3 1	7
1 5 3 8 4 6	
1, 5 4 4 3 7 7	9
1 5 3 8 4 6	
1, 6 9 8 2 2 3	11
1 5 3 8 4 6	
1, 8 5 2 0 6 9	$26 \equiv 0 \pmod{13}$
1 5 3 8 4 6	
2, 0 0 5 9 1 5	

The period-number of $1/13^2$ is obtained by writing down the results of the successive additions. Where the sums have the additional digit 1 or 2 (separated by a comma) these are added to the last digit of the previous sum. Thus the period of $1/13^2$ is

$\dot{0}$ 0 5 9 1 7
1 5 9 7 6 3
3 1 3 6 0 9
4 6 7 4 5 5
6 2 1 3 0 1
7 7 5 1 4 7
9 2 8 9 9 4
0 8 2 8 4 0
2 3 6 6 8 6
3 9 0 5 3 2
5 4 4 3 7 8
6 9 8 2 2 4
8 5 2 0 7 $\dot{1}$

For $1/13^3$ the number of digits will be $6 \times 13^2 = 1014$ which we obtain, as before, in 13 sets of 78 digits. The first set is obtained by dividing the P. N. of $1/13^2$ say r' by 13. The remainder will be, according to § 3 the same as before, namely 2. So the next set of 78

digits is obtained by adding $2 \times 076923 = 153846$ to every successive group of 6 digits in the first set. The first 2 sets are given below :

First set		*Common difference*		*Second set*
0 0 0 4 5 5		1 5 3 8 4 6		1 5 4 3 0 1
1 6 6 1 3 5		1 5 3 8 4 6		3 1 9 9 8 1
6 3 9 5 0 8		1 5 3 8 4 6		7 9 3 3 5 4
4 2 0 5 7 3		1 5 3 8 4 6		5 7 4 4 1 9
5 0 9 3 3 0		1 5 3 8 4 6		6 6 3 1 7 6
9 0 5 7 8 0	+	1 5 3 8 4 6	=	1, 0 5 9 6 2 6
6 0 9 9 2 2		1 5 3 8 4 6		7 6 3 7 6 8
6 2 1 7 5 6		1 5 3 8 4 6		7 7 5 6 0 2
9 4 1 2 8 3		1 5 3 8 4 6		1, 0 9 5 1 2 9
5 6 8 5 0 2		1 5 3 8 4 6		7 2 2 3 4 8
5 0 3 4 1 3		1 5 3 8 4 6		6 5 7 2 5 9
7 4 6 0 1 7		1 5 3 8 4 6		8 9 9 8 6 3
2 9 6 3 1 3 Rem. 2		1 5 3 8 4 6		4 5 0 1 5 9 Rem. 4

The 1 preceding the comma should be added to the last digit of the previous row.

The 3rd and other sets may be obtained similarly. Thus, given P. N. of $1/d^n$ we can find P. N. of $1/d^{n+1}$ by adding successively to it a common difference obtained as follows:—

Multiply P. N. of $1/d$ by the remainder when it is divided by d and repeat this d^{n-1} times as in the above example.

Mayavaram.

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REVIEWS

M. RAZIUDDIN SIDDIQI, M.A. (Cantab), Ph.D. (Leipzig). *Lectures on Quantum Mechanics.* Osmania University Publications, 1938 pp. 293.

This book gives an introductory account of old and new quantum mechanics and their elementary applications. After a brief survey of the essentials of the special theory of Relativity and the general dynamical theory with its Lagrangian and Hamiltonian functions, angle and action variables, canonical equations and contact transformations, the book proceeds to describe the Rutherford model of the atom followed by an account of the Quantum theory of radiation illustrating its particle nature by a consideration of the photoelectric and the Compton effects. Then Bohr's theory of the atom, the correspondence principle and Sommerfeld's relativity theory of the fine structure of spectral lines come up for detailed treatment.

A criticism of Bohr's theory and Heisenberg's quantum mechanics with a detailed account of matrix calculation, Poisson bracket, and angular momentum, illustrated by application to the harmonic oscillator, are dealt with in the succeeding chapter. This is followed by a description of de Broglie's wave idea of matter and Schrödinger's derivation of the differential equation for matter waves. The polynomial method of determining the *eigen* values with applications to the harmonic oscillator and the hydrogen-like atoms accompanied by consideration of the equivalence of the methods of Heisenberg and Schrödinger and the derivation of the matrix elements from *eigen* functions is clearly treated in the next chapter. Perturbation theory of the first and second orders for non-degenerate and degenerate cases with the Stark effect and the Helium atom as examples and the same theory in Heisenberg's mechanics are then considered in detail. The physical interpretation of the wave functions by Schrödinger and Born, Heisenberg's principle of indeterminancy relativistic quantum mechanics and Dirac's theory of the electron form the subject-matter of the closing chapters of the book.

Almost all the above topics have been dealt with before in numerous books on quantum mechanics. The particular value of this book lies in its abundance of expository detail. Many students of Physics are quite unfamiliar with the mathematical technique required for a proper understanding of the principles and applications of quantum mechanics. The required mathematical apparatus is almost completely given in the relevant portions of the book. Perhaps a brief account of Legendre and Laguerre polynomials and the different systems of orthogonal co-ordinates used in quantum mechanics may not be out of place in this book.

There are unfortunately quite a number of misprints which a more careful proof correction could have avoided. References to previous articles and equations are also not given correctly in several places.

The book is intended for readers with no extensive training in higher mathematics and is written with a good appreciation of the requirements of such readers. Clearness in exposition is attained by full explanations of the various steps in almost all the difficult mathematical calculations. The book is an excellent guide to those Physicists and Chemists, who desire to have a working knowledge of quantum mechanics. The students who read this book will be eagerly awaiting the publication of the second volume promised by the author in his preface.

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G. S. MAHAJANI: *The application of moving axes methods to the Geometry of Curves and Surfaces*, 1937, Aryabhushan Press, Poona, 60 pp.

Students of Mathematics (in which class I include firstly the teachers of the subject !), for whom the language barrier has stood in the way of an appreciation of the beautiful methods associated with the *trièdre mobile*, will be grateful to Dr. Mahajani for this elementary introduction to the subject.

After a brief explanation of the vectorial notation, the author obtains in the first chapter the fundamental formulae connecting the rates of change of a vector referred to fixed and to moving frames of reference.

In the second chapter of 26 pages we have a fairly complete account of the better known differential properties of space curves *viz.*, Frenet's Formulae, the developables associated with a curve, spherical curvature, Bertrand Curves etc. While most English text-books give a left-handed treatment of the subject, the author uses right-handed reference frames (equivalent to a reversal of the sense of the binormal) and obtains as the reward of his chivalry a more symmetric distribution of signs in Frenet's Formulae.

The third chapter deals briefly with the frames associated with a surface and includes Meunier's and Euler's Theorems, geodesic curvature and torsion, lines of curvature, Asymptotic lines and Joachimsthal's Theorem. Chapter IV which is devoted to applications deals with curves on the sphere, the cone and the cylinder. There is a collection of 10 examples given at the end while there are a few given in the body of the book, mostly taken from Tripos papers.

The book is apparently not intended to be a text-book on Differential Geometry; for how can one deal adequately with the geometry of "Curves on surfaces" in a ten-page chapter? However, to the student who is already acquainted with the main results of the subject by the usual methods, Dr. Mahajani's book offers the stimulating experience of traversing the same field in a new vehicle and of recognising the same landmarks from a new perspective. The author's clarity of exposition and helpful remarks ensure that the reader's vision of the domain from his moving axes shall be bright and clear.

A. NARASINGA RAO.

SOLUTIONS TO QUESTIONS

Question 1654

(S. CHOWLA):—Given any $\alpha > 0$, is it true that the numbers $\log(1+p^{-\alpha})$ where p runs through a finite number of primes are linearly independent *i.e.* there is no relation of the form

$$\sum c_p \log(1+p^{-\alpha}) = 0$$

where the numbers c_p are integers?

Prove this result in the special case when α is a positive rational number $\leqslant 1$.

Solution by Proposer

Let $\alpha = c/d$ where $(c, d) = 1$. Let

$$\prod_p (1+p^{-c/d})^{c_p} = 1 \quad \text{or} \quad \prod_p (p^{c/d}+1)^{c_p} = \prod_p (p^{c_p \cdot c/d}). \qquad \ldots (1)$$

Raising both sides to the power d, we easily see that (1) is impossible—the left side is irrational, the right is rational. The case $\alpha = 1$ needs separate treatment.

Question 1686

(A. A. KRISHNASWAMI AYYANGAR):—If $\int_a^b x^n f(x) = 0$ for integral $n \geqslant 0$, is $f(x)$ identically zero?

Solution by the proposer

If (a, b) is a finite interval, $f(x)$ is zero except for a set of measure zero and is identically zero if $f(x)$ is continuous. If (a, b) is an infinite interval, the result is no longer true even if $f(x)$ be continuous. Examples, in this connection, are:—

(i) $\int_0^\infty x^n [\exp(x^\mu \cos \mu\pi)] \sin(x^\mu \sin \mu\pi) = 0$ for $n = 0, 1, 2, \cdots$ and $0 < \mu < 1/2$.

(ii) $$\int_0^\infty x^n \left[\exp\left(-\frac{\pi\sqrt{x} - \log x}{(\log x)^2 + \pi^2}\right)\right] \sin \frac{\sqrt{\pi}\log x + \pi}{\pi^2 + (\log x)^2}\, dx = 0.$$

(For these examples see Polya and Szego, *Aufgahen and Lehrsatze* Abschitt 2, Questions 138, 139, 142. A simpler example is to be found in Titahmarsh, *Theory of Functions*, p. 106-107.)

(iii) $\int_0^\infty x^n \exp(-x^{\frac{1}{4}}) \sin x^{\frac{1}{4}} = 0, \qquad n = 0, 1, 2, 3, \cdots$

Question 1707

(A. A. KRISHNASWAMI AYYANGAR):—A, A′ denote two integers such that the order of the digits in one is reversed in the other. Investigate the existence of numbers which can be represented in more than one way as the product of two such members A and A′. Eg. $3472 \times 2743 = 1456 \times 6541$.

Solution by M. Sitaramanjanujalu

I give below a general rule for obtaining pairs of nos. such that $AA' = BB'$.

Let x and y be two nos. each with any no. of digits. Let x' and y' be the nos. formed by reversing the digits in x and y respectively. The nos. x and y should be such that they satisfy the following conditions.

(i) Any digit in x when multiplied by any digit in y should not give a number greater than 9.

(ii) When x is multiplied by y or y' as in the ordinary multiplication in Arithmetic (digit by digit as shown below), the sum of the digits in any column should not exceed 9. (This includes indirectly case (i) also).

Then $(x \cdot y) \times (x' \cdot y') = (x \cdot y') \times (x' \cdot y)$

xy and $x'y'$ give a pair of nos. like A + A′ and $x \cdot y'$ and $x' \cdot y$ give another pair.

For example

$$(21121 \times 1112)(12112 \times 2111) = (21121 \times 2111)(12112 \times 1112)$$

or $$23486552 \times 25568432 = 44586431 \times 13468544$$

Actual multiplication as given below makes the above reasoning clear.

(x) 21121 (y) 1112	(x') 12112 (y') 2111	(x) 21121 (y') 2111	(x') 12112 (y) 1112
42242 21121 21121 21121	24224 12112 12112 12112	42242 21121 21121 21121	24224 12112 12112 12112
23486552	25568432	44586431	13468544

N. B.—The case when $x = x'$ or $y = y'$ is trivial.

e.g., $132 \times 231 = 131 \times 231$.

Question 1715

(A. A. KRISHNASWAMY AYYANGAR) :—When two integers are multiplied together, show that the digit in the highest place in the product can never be intermediate between the corresponding digits in the given factors.

Solution by A. Ranganatha Rao

Let F_1, F_2 be the factors and P the product, f_1, f_2, p the digits in their highest places and n_1, n_2, m the number of digits in F_1, F_2, P, respectively. Then it is easy to see that $m = n_1 + n_2$ or $n_1 + n_2 - 1$. We exclude the case when $p = f_1$ or f_2, for then, nothing remains to be proved. We visualise the process of division to be actually performed with P as the dividend and F_1, say, as the divisor, so that F_2 is the quotient. Considering the very first and the subsequent stages of the division, it is clear that the case $m = n_1 + n_2$ can arise only if $p < f_1$. Then, similarly regarding F_2 instead of F_1 as the divisor, p must be less than f_2 also. Again, it is shown in a similar way that the case $m = n_1 + n_2 - 1$ can arise only if $p > f_1$ and f_2. Hence the result.

Question 1739

(A. A. KRISHNASWAMY AYYANGAR) :—Show that

$\int_0^x e^{-x^2/2}dx = \sqrt{\frac{\pi}{2}}\left(1 - e^{-mx^2}\right)$ when $m > \frac{1}{2}$ has only one real root in the interval $(0, \infty)$. If α_m is the root, prove that

(i) $\alpha_m \to 0$ as $m \to \infty$

(ii) $\int_0^x e^{-x^2/2}dx > \sqrt{\frac{\pi}{2}}\left(1 - e^{-mx^2}\right)$ in $0 < x < \alpha_m$

and (iii) $\int_0^x e^{-x^2/2}dx < \sqrt{\frac{\pi}{2}}\left(1 - e^{-mx^2}\right)$ when $x > \alpha_m$.

Additional Solution by N. Vencoba Rao

Let $f(x) = \int_0^x e^{-x^2/2}dx - \sqrt{\frac{\pi}{2}}\left(1 - e^{-mx^2}\right)$

Then $f'(x) = e^{-mx^2}\left[e^{(m-1/2)x^2} - \sqrt{2\pi} \cdot mx\right]$

Let $\phi(x) = e^{(m-1/2)x^2} - \sqrt{2\pi} \cdot mx$; so that

$$\phi'(x) = e^{(m-1/2)x^2}\, 2(m-1/2)x - m\sqrt{2\pi}$$

$$\therefore \quad \phi''(x) = 2 \cdot (m - 1/2)\ e^{(m-1/2)x^2} \left\{ 1 + 2\ (m - 1/2)x^2 \right\}$$

which for $m > 1/2$ is always positive.

So $\phi'(x)$ is always increasing and hence has only one root β. (i.e.) $\phi'(x)$ is negative in $(-\infty, \beta)$ and positive in (β, ∞).

It can be verified that $\phi'(1/\sqrt{2m-1})$ is negative and so $\beta > 1/(\sqrt{2m-1})$; but β is given by $e^{(m-1/2)\beta^2} \cdot 2\ (m-1/2)\ \beta = m\sqrt{2\pi}$

$\therefore\ \phi\ (\beta) = e^{(m-1/2)\beta^2} - e^{(m-1/2)\beta^2} \cdot (2m-1)\beta^2$ which when $\beta^2 > 1/(2m-1)$ and $m > 1/2$ is negative and $\phi\ (0)$ is positive.

So $\phi\ (x)$ and hence $f'\ (x)$ is decreasing in $(0, \beta)$ and increasing in (β, ∞); also $f'\ (\beta)$ is negative. So $f'\ (x)$ has two and only two real roots say (λ, μ).

It follows that $f(x)$ is increasing in $(0, \lambda)$, decreasing in (λ, μ) and increasing in (μ, ∞). The graph of $f(x)$ in $(0, \infty)$, rises from the origin, to a certain maximum value at $x = \lambda$, descends in (λ, μ) and again rises asymptotically towards the x-axis. Since $f(\infty) = 0$, and $f(x)$ is increasing in (μ, ∞) $f(\mu)$ is negative. So $f(x)$ vanishes once in (λ, μ) say at α_m. The results (ii) and (iii) of question 1739 now follow.

(i) λ and μ are given by $e^{(m-1/2)x^2} = mx\ \sqrt{2\pi}$.

Taking logarithms of both sides it follows that

$$x^2 = \frac{\log\sqrt{2\pi}}{m-\frac{1}{2}} + \frac{\log\ m}{m-\frac{1}{2}} + \frac{\log\ x}{m-\frac{1}{2}}$$

and hence $\lim_{m\to\infty} x^2 = 0$; i.e. as $m \to \infty$, λ and $\mu \to 0$ and so $\alpha_m \to 0$.

Question 1755

(B. R. VENKATARAMAN):—The necessary and sufficient condition that the F and ϕ conics of two conics S, S′ be concentric is that S and S′ be concentric. In this case the four conics have the same centre and the asymptotes of the four conics belong to the same involution pencil.

Solution by the Proposer and A. Ranganatha Rao.

It is known that all the four conics in question have a common self-polar triangle. If any two of the conics are concentric, the common centre and the line at infinity constitute a vertex and the

opposite side of the common self-polar triangle of these two conics, and therefore of all the four conics. Again, the remaining two sides of the self-polar triangle are conjugate diameters for each of the conics; hence these lines separate harmonically the asymptotes of the four conics which must, therefore, belong to an involution pencil.

We may also note that since two parabolas with parallel axes are concentric conics their F and ϕ conics are also parabolas whose axes are parallel to those of the given parabolas.

QUESTIONS FOR SOLUTION

1780. (M. S. SRINIVASAN):—If p be prime and $M = 1\cdot 3\cdot 5\cdot 7 \cdot\cdot (p-2)$ prove that $M^4 \equiv 1 \pmod{p}$.

1781. (P. KESAVA MENON):—Show that

$$\log (n+1) - e \sum^{n} \{r^r/(r+1)^{r+1}\}$$

tends to a definite limit which lies between zero and $2(1-\log 2)-\gamma$ when $n \to \infty$, γ being Euler's constant.

1782. (P. KESAVA MENON):—If P denotes the perimeter of the ellipse whose semi axes are a and b respectively, show that

$$\pi \sqrt{2a^2+2b^2} > P > \pi (a+b)$$

and hence deduce that when a and b are very nearly equal, the perimeter may be taken to be $\pi(a+b)$.

1783. (A. NARASINGA RAO):—From a projective space P_n of n dimensions, an open set of points P'_n is obtained by removing all points on a straight line in P_n. What are the topological properties of the set P'_n?

1784. (D. KRISHNA RAO):—Show that

$$\iiiiiint \sum \delta p_i \; \delta q_i \; \delta p_j \; \delta q_j \; \delta p_k \; \delta q_k$$

where the summation is extended over the $n(n-1)(n-2)/6$ combinations of the indices i, j, k is an integral invariant of any Hamiltonian system in which q_i, p_i are the variables.

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